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Involutive symmetries, supersymmetries and reductions of the Dirac equation

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Abstract. A new algebra of involutive symmetries of the Dirac equation is found. This algebra is used to reduce the Dirac equation for a charged particle, interacting with an external field and to describe hidden supersymmetries of this equation. Reducibility of a class of equations of supersymmetric quantum mechanics is established.

1. Introduction

It is well known that symmetries of differential equations form powerful tools for the study of these equations. They are used to separate variables [1], to derive conservation laws [2], to construct exact solutions of linear and nonlinear differential equations [3–6], to find spectra of linear differential operators [7, 8], and so on.

In this paper we investigate special involutive symmetries of the Dirac equation. It is well known that this equation is invariant with respect to the extended Poincaré group. Pauli, Gürsey, Plebanski and Pursey [9] found the additional $SL(2, C)$ symmetry of the Dirac equation, which is realized by antilinear transformations (i.e., including the complex conjugation). Hidden $SL(2, C)$ symmetry of this equation (generated by linear non-local integro-differential operators and by first-order differential operators with matrix coefficients) was described in [10] and [11] (refer also to [8]).

In this paper we present a new symmetry algebra of the Dirac equation. It is specified by the following features.

(i) All its basis element are involutions.

(ii) It includes proper discrete symmetries (like reflections P, T and charge conjugation C) as well as finite rotations.

(iii) It is a finite-dimensional Lie algebra whose dimension is much more extended than dimensions of other finite symmetry algebras of the Dirac equation.

We use this symmetry algebra for two purposes. First, to reduce the Dirac equation to two uncoupled subsystems or even to four uncoupled one-component equations. The necessary and sufficient condition for existence of such a reduction is that the components of the vector potential A_μ (treated as given functions of x_0, x_1, x_2, x_3) have definite parities, i.e., are invariant (up to a sign) under reflections of x_μ.

The other important application of involutive symmetries is searching for systems with exact supersymmetry (SUSY). Using the former algebra we extend the list of known systems

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with $N = 2$ SUSY [12, 13] and find a class of external potentials for the Dirac equation which generated extended SUSY.

In section 2 we describe the involutive symmetry algebra of the Dirac equation. The corresponding reductions for the Dirac equation are discussed in section 3 and are presented explicitly in the appendix.

Sections 4 and 5 are devoted to reduction of the Dirac oscillator and to searching for exact SUSY. Section 6 includes application of the reduction technique to SUSY quantum mechanics.

2. Involutive symmetries of the Dirac equation

We start with the free Dirac equation

$$L_0\psi \equiv (\gamma^\mu p_\mu - m)\psi = 0 \tag{2.1}$$

which is invariant with respect to the complete Lorentz group. Here γ_μ ($\mu = 0, 1, 2, 3$) are the Dirac matrices with diagonal $\gamma_5 = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$:

$$\gamma_0 = \begin{pmatrix} 0 & \mathbf{I}_2 \\ \mathbf{I}_2 & 0 \end{pmatrix} \qquad \gamma_a = \begin{pmatrix} 0 & -\boldsymbol{\sigma}_a \\ \boldsymbol{\sigma}_a & 0 \end{pmatrix} \qquad \gamma_5 = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \mathbf{I}_2 & 0 \\ 0 & -\mathbf{I}_2 \end{pmatrix} \tag{2.2}$$

σ_a, $a = 1, 2, 3$, are the Pauli matrices and $\mathbf{I}_2$ denotes a 2×2 unit matrix.

Let us note that this equation admits also non-Lie symmetries [8, 10, 11].

Here we study the class of involutive symmetries of (2.1). Such symmetries form a subset of the complete Lorentz group, which is defined by reflections of coordinate axes, rotations by the angle π with respect to a given axis (each of them can be reduced to a reflection of a pair of axes) and by products of these transformations. There are 16 of them, and they form a finite group composed of:

- four reflections of coordinates x_μ,

$$x_\mu \rightarrow (\theta_\lambda x)_\mu = (1 - 2\delta_{\lambda\mu})x_\mu \qquad \lambda = 0, 1, 2, 3 \tag{2.3a}$$

- six reflections of pairs of coordinates,

$$x_\mu \rightarrow (\theta_{\lambda\sigma} x)_\mu = (1 - 2\delta_{\lambda\mu} - 2\delta_{\sigma\mu})x_\mu \qquad \lambda \neq \sigma \qquad \lambda, \sigma = 0, 1, 2, 3 \tag{2.3b}$$

- four reflections of triplets of coordinates,

$$x_\mu \rightarrow (\theta'_\lambda x)_\mu = (2\delta_{\lambda\mu} - 1)x_\mu \qquad \lambda = 0, 1, 2, 3 \tag{2.3c}$$

- a complete reflection of all coordinates,

$$x_\mu \rightarrow (\theta x)_\mu = -x_\mu \tag{2.4a}$$

- and the identity transformation

$$x_\mu \rightarrow (I x_\mu) = x_\mu \tag{2.4b}$$

($\mu = 0, 1, 2, 3$; no sums over μ in (2.3)). We will also use the following notation for reflections (2.3) and (2.4):

$$\theta = \theta_{54} \qquad \theta_\nu = \theta_{5\nu} \qquad \theta'_\nu = \theta_{4\nu}. \tag{2.5}$$

We see that for $\lambda, \sigma = 1, 2, 3$ transformations (2.3*b*) are rotations while (2.3*a*), (2.3*c*) and (2.3*b*) for zero λ or σ are proper reflections.

The corresponding symmetries of the Dirac equation form a projective representation [14] of the 16-dimensional group (2.3) and (2.4), and have the following form

$$\psi(x) \rightarrow R_{kl}\psi(x) = \hat{S}_{kl}\hat{\theta}_{kl}\psi(x) \equiv \hat{S}_{kl}\psi(\theta_{kl}x) \tag{2.6}$$

where

$$\hat{S}_{\mu\nu} = \tilde{\gamma}_\mu \tilde{\gamma}_\nu \qquad \hat{S}_{5\mu} = \tilde{\gamma}_5 \tilde{\gamma}_\mu \qquad \hat{S}_{4\mu} = -\hat{S}_{\mu 4} = \gamma_\mu$$
$$\tilde{\gamma}_0 = \gamma_0 \qquad \tilde{\gamma}_5 = \gamma_5 \qquad \tilde{\gamma}_a = \mathrm{i}\gamma_a$$
$$k, l, = 0, 1, \ldots 5 \qquad m = 0, 1, 2, 3 \qquad a = 1, 2, 3$$

and the trivial identity transformation (corresponding to (2.4*b*) is omitted.

It is easy to verify that such defined operators R_{kl} commute with the operator L_0 and so transform solutions of (2.1) into themselves.

Symmetries (2.6) satisfy the following commutation relations

$$[R_{kl}, R_{mn}] = 2\mathrm{i}(\delta_{kn} R_{lm} + \delta_{lm} R_{kn} - \delta_{ln} R_{km} - \delta_{km} R_{ln}) \tag{2.7}$$

(by definition $R_{ab} = -R_{ba}$).

In accordance with (2.7) symmetries (2.6) realize a representation of the algebra $so(6)$.

Let us now specify antilinear (i.e. including the complex conjugation) symmetries of equation (2.1) corresponding to reflections (2.3) and (2.4). On the set of solutions of the Dirac equation they are reduced to the form

$$\psi(x) \to B_{kl}\psi(x) \equiv C R_{kl}\psi(x) \tag{2.8}$$

where R_{kl} are transformations (2.6) and C is the charge conjugation transformation

$$C\psi(x) = \gamma_2 c\psi(x) \equiv \mathrm{i}\gamma_2\psi^*(x). \tag{2.9}$$

Using the relations

$$[C, R_{\lambda a}] = \{C, R_{\lambda\sigma}\} = \{C, R_{ab}\} = 0$$
$$a, b, c = 1, 2, 3 \qquad \lambda, \sigma = 0, 4, 5 \tag{2.10}$$

we conclude that among the transformations (2.7) there are six representatives which satisfy $(B_{AC})^2 = -I$ (for $A, C = 0, 4, 5$ or $A, C = 1, 2, 3$) and nine representatives which satisfy the condition $(B_{AC})^2 = I$, where I is the identity operator. We have a special interest in such transformations (2.7) whose square is positive (otherwise the corresponding $B_{\mu\lambda}$ cannot be diagonalized to the real matrix γ_5 and so they cannot be used for reductions considered in the following section). The corresponding symmetries are

$$B_{4a} = C R_{4a} \tag{2.11a}$$
$$B_{5a} = C R_{5a} \tag{2.11b}$$
$$B_{0a} = C R_{0a}. \tag{2.11c}$$

Using (2.7) and (2.10) it is not difficult to specify commutation and anticommutation relations for operators (2.6) and (2.11). We notice that the set of operators $\{\hat{R}_{kl} = \mathrm{i}R_{kl}, B_{\alpha a}, C\}$ forms a basis of the 25-dimensional Lie algebra A_{25} characterized by commutation relations (2.7) and (2.12):

$$\begin{aligned}
&[B_{\alpha a}, B_{\beta b}] = -2(\delta_{ab}\hat{R}_{\alpha\beta} + \delta_{\alpha\beta}\hat{R}_{ab})\\
&[B_{\alpha a}, \hat{R}_{\beta\sigma}] = 2(\delta_{\alpha\beta} B_{\sigma a} - \delta_{\alpha\sigma} B_{\beta a})\\
&[B_{\alpha a}, \hat{R}_{bc}] = -2(\delta_{ac} B_{\alpha b} - \delta_{ab} B_{\alpha c})\\
&[B_{\alpha a}, \hat{R}_{\beta b}] = \epsilon_{a\alpha b\beta c\sigma} B_{\sigma c} - 2\delta_{ab}\delta_{\alpha\beta} C\\
&[C, B_{\alpha a}] = 2\hat{R}_{\alpha a} \qquad [C, \hat{R}_{\alpha a}] = 2B_{\alpha a}\\
&[C, \hat{R}_{ab}] = [C, \hat{R}_{\alpha\beta}] = 0\\
&a, b, c = 1, 2, 3 \qquad \alpha, \beta, \sigma = 0, 4, 5.
\end{aligned} \tag{2.12}$$

Remark. By including all symmetries (2.8) and products of symmetries (2.6) and (2.8), and the operator of multiplication $\mathrm{i} = \sqrt{-1}$, the algebra A_{25} can be extended to the 64-dimensional Lie algebra defined over the field of real numbers. Additional extensions can be made by including the non-Lie involutive symmetries [8, 11].

Thus involutive symmetries of the Dirac equation generate the extended Lie algebra A_{25}. In the following sections we use it to reduce the Dirac equation for a charged particle interacting with various external fields and to search for supersymmetries of the Dirac equation.

3. Reduction of the Dirac equation

Now we shall apply the results of the previous subsection to reduce the Dirac equation for a charged particle in an external field

$$L\psi \equiv (\gamma^{\mu}\pi_{\mu} - m)\psi = 0 \tag{3.1}$$

where $\pi_{\mu} = p_{\mu} - eA_{\mu}$, $p_{\mu} = \mathrm{i}\partial/\partial x^{\mu}$, $A_{\mu} = A_{\mu}(x) = A_{\mu}(x_0, \boldsymbol{x})$ is the vector potential.

Equation (3.1) is invariant under one of the transformations described in (2.4*a*) and (2.3*a*–*c*) provided the vector potential A_{μ} satisfies one of the relations,

$$A_{\mu}(-x_0, -\boldsymbol{x}) = -A_{\mu}(x_0, \boldsymbol{x}) \tag{3.2}$$

$$A_{\mu}(\theta_{\lambda}x) = (1 - 2\delta_{\mu\lambda})A_{\mu}(x) \tag{3.3a}$$

$$A_{\mu}(\theta_{\lambda\sigma}x) = (1 - 2\delta_{\mu\lambda} - 2\delta_{\mu\sigma})A_{\mu}(x) \tag{3.3b}$$

$$A_{\mu}(\theta'_{\lambda}x) = (2\delta_{\mu\lambda} - 1)A_{\mu}(x) \tag{3.3c}$$

respectively, with λ and σ being fixed. On the other hand, if we require that (3.1) admits one of the symmetries (2.11*a*–*c*), the vector potential A_{μ} has to satisfy one of the corresponding relations,

$$A_{\mu}(\theta'_{a}x) = -(2\delta_{\mu a} - 1)A_{\mu}(x) \tag{3.4a}$$

$$A_{\mu}(\theta_{a}x) = -(1 - 2\delta_{\mu a})A_{\mu}(x) \tag{3.4b}$$

$$A_{\mu}(\theta_{0a}x) = -(1 - 2\delta_{\mu 0} - 2\delta_{\mu a})A_{\mu}(x) \tag{3.4c}$$

respectively.

We note that relations (3.3) and (3.4) leave the Lorentz gauge $\partial_{\mu}A^{\mu} = 0$ invariant.

To reduce (3.1) we diagonalize the corresponding symmetries (2.6). Let us consider in detail the case (3.2), i.e. when equation (3.1) is invariant under the transformation

$$\hat{R}\psi(x) = \gamma_5\hat{\theta}\psi(x) = \gamma_5\psi(-x). \tag{3.5}$$

To diagonalize this symmetry we use the operator

$$W = \frac{1}{\sqrt{2}}(1 + \gamma_5\gamma_0)\frac{1}{\sqrt{2}}(1 + \gamma_5\gamma_0\hat{\theta}) = \hat{\theta}_+ + \gamma_0\gamma_5\hat{\theta}_- \tag{3.6}$$

with $\hat{\theta}_{\pm} = \frac{1}{2}(1 \pm \hat{\theta})$, then

$$W\hat{R}W^{\dagger} \equiv W\gamma_5\theta W^{\dagger} = \gamma_5. \tag{3.7}$$

Simultaneously, the operator L of (3.1) is reduced to the block diagonal form:

$$WLW^{\dagger} = L' = -\gamma_5\pi_0 - \tfrac{1}{2}\mathrm{i}\epsilon_{abc}\gamma_a\gamma_b\pi_c\hat{\theta} - m. \tag{3.8}$$

Thus the transformed equation

$$L'\psi' = 0 \qquad \psi' = U\psi \tag{3.9}$$

has the desired reduced form

$$(-\mu\pi_0 - \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\hat{\theta} - m)\psi'_\mu = 0 \qquad \mu = \pm 1 \tag{3.10}$$

where ψ'_μ are two-component spinors, i.e. non-zero components of eigenvectors of γ_5 satisfying $\gamma_5\psi' = \mu\psi'$.

For $A_\mu = 0$ equation (3.10) is equivalent to the one considered in [15].

If equations (3.10) again admit a discrete symmetry, say

$$\psi_\mu(x) \to \hat{R}\psi_\mu = \sigma_3\hat{\theta}_{12}\psi_\mu(x) \equiv \sigma_3\psi_\mu(x') \qquad x' = (\theta_{12}x) \tag{3.11}$$

(which is the case of $A_0(x') = A_0(x)$, $A_1(x') = -A_1(x')$, $A_2(x') = -A_2(x)$, $A_3(x') = A_3(x)$), then they can further be reduced to one-component uncoupled subsystems. Indeed, by diagonalizing symmetry $\hat{R} = \sigma_3\hat{\theta}_{12}$ and using the transformation $\psi'_\mu \to \psi''_\mu = W\psi'_\mu$ (the corresponding transformation operator is $W = [1 - \hat{\theta}_{12} - \mathrm{i}\sigma_2(1 + \hat{\theta}_{12})]/2$, $W^{-1} = W^\dagger$) we change equation (3.10) to the following,

$$[-\mu\pi_0 - \lambda\pi_1 - \hat{\theta}_{12}(\pi_3 + \mathrm{i}\pi_2) - m]\psi_{\mu\lambda} = 0 \tag{3.12}$$

where both μ and λ runs independently over the values $+, -$ and $\psi_{\mu\lambda}$ are one-component functions, i.e. non-zero components of eigenvectors of matrix σ_3 satisfying $\sigma_3\psi''_\mu = \lambda\psi''_\mu$.

We notice that transformations (3.6), (3.7) and (3.8) can also be used for the reduction of Dirac's equation with the anomalous (Pauli) interaction:

$$\left(\gamma^\mu\pi_\mu - m - \frac{ke}{2m}S_{\mu\nu}F^{\mu\nu}\right)\psi = 0 \tag{3.13}$$

where

$$S_{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma_\mu, \gamma_\nu] \qquad F_{\mu\nu} = \mathrm{i}[\pi_\mu, \pi_\nu].$$

For example, if A_μ satisfies (2.4*a*), then (3.13) is reduced to the following two subsystems for two-component spinors:

$$\left(-\mu\pi_0 - \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\hat{\theta} + \frac{ek}{m}\boldsymbol{\sigma}\cdot\boldsymbol{H} - \mu\frac{\mathrm{i}ek}{m}\boldsymbol{\sigma}\cdot\boldsymbol{E}\hat{\theta} - m\right)\psi_\mu = 0 \qquad \mu = \pm 1$$

where $\boldsymbol{H}$ and $\boldsymbol{E}$ are vectors of the magnetic and electric field strengths: $E_a = F_{0a}$, $H_a = \epsilon_{abc}F_{bc}/2$.

In an analogous way it is possible to reduce the Dirac equation (3.1) if vector potentials satisfy one of the relations (3.3) or (3.4). We present the complete list of the corresponding reductions in the appendix.

4. Reduction of the Dirac oscillator

The Dirac oscillator equation [17, 18] can be written as

$$(\gamma^\mu p_\mu - \mathrm{i}\omega\gamma_0\gamma_a x_a - m)\psi = 0. \tag{4.1}$$

This equation is P-invariant, i.e. admits the following involutive symmetry,

$$x_0 \to x'_0 = x_0 \qquad \boldsymbol{x} \to \boldsymbol{x}' = -\boldsymbol{x} \qquad \psi(x_0, \boldsymbol{x}) \to \gamma_0\psi(x_0, -\boldsymbol{x})$$

and consequently can be reduced to two uncoupled subsystems. Indeed, using the transformation $\psi \to \psi' = W\psi$ where $W = (1/\sqrt{2})(1 - \mathrm{i}\gamma_5 R_{40})$, equation (4.1) decouples and can be expressed as

$$p_0\psi_\pm = [\pm(\boldsymbol{\sigma}\cdot\boldsymbol{p} + m\hat{\theta}_0') + \mathrm{i}\omega\boldsymbol{\sigma}\cdot\boldsymbol{x}\hat{\theta}_0']\psi_\pm. \tag{4.2}$$

Equations (4.2) admit involutive symmetry (3.9) and by means of the operator W for (3.9) can be reduced to four one-component uncoupled equations.

However, there is another involutive symmetry for equation (4.2) which can be written as

$$Q = B\hat{\theta}_0' \qquad Q^2 \equiv 1 \tag{4.3}$$

where B is the Biedenharn operator [18]:

$$B = \frac{q}{|q|} \qquad q = \boldsymbol{\sigma}\cdot\boldsymbol{L} + 1 \qquad \boldsymbol{L} = \boldsymbol{x}\times\boldsymbol{p}. \tag{4.4}$$

Operator B anticommutes with $\sigma\cdot\boldsymbol{p}$ and $\sigma\cdot\boldsymbol{x}$; thus Q introduced in (4.3) commutes with the operator in square brackets defined in (4.2). On the set of functions ψ_ρ^μ satisfying

$$Q\psi_\rho^\mu = \mu\psi_\rho^\mu \qquad \rho = \pm \qquad \mu = \pm 1 \tag{4.5}$$

equations (4.2) are reduced to the form

$$p_0\psi_\rho^\mu = (\rho\boldsymbol{\sigma}\cdot\boldsymbol{p} + \mu\rho m B + \mathrm{i}\mu\boldsymbol{\sigma}\cdot\boldsymbol{x}B)\psi_\rho^\mu. \tag{4.6}$$

In other words, the Dirac oscillator equation is reduced to four uncoupled two-component subsystems.

Setting $m = 0$ in (4.2) and (4.6), we receive the equations which we shall call the Weyl oscillators. Analogously to the Dirac oscillator case they generate oscillator-like spectra and are related to the free (Weyl) equation by changing $\boldsymbol{p} \to \boldsymbol{p} - \mathrm{i}\omega\boldsymbol{x}\beta$ with $\beta = P$ or $\beta = B$ being an operator anticommuting with the differential part of the corresponding Hamiltonian in (4.2) or (4.6), respectively.

The Weyl oscillators will be studied in more detail in the next paper. It appears they have very interesting symmetries and supersymmetries which are preserved if we change $\boldsymbol{x} \to \partial W/\partial\boldsymbol{x}$ in (4.6), where $W(x)$ is an arbitrary even superpotential.

5. Extended supersymmetries

We say that an equation of motion is supersymmetric if it admits specific symmetries (supercharges) Q_a, $a = 1, 2$, which form the Witten superalgebra [19] (we choose Q_a Hermitian):

$$\{Q_a, Q_b\} = 2\delta_{ab}\hat{H} \qquad [Q_a, \hat{H}] = 0 \tag{5.1}$$

$\hat{H}$ is the related Hamiltonian.

To search for SUSY we use the following anticommutative relations for involutive symmetries (2.6):

$$\{R_{kl}, R_{mn}\} = \epsilon_{klmngf}R_{fg} + 2\delta_{km}\delta_{ln}I - 2\delta_{lm}\delta_{kn}I. \tag{5.2}$$

We start with the Dirac equation (3.1) for a charged particle interacting with the time-independent magnetic field. The corresponding vector potentials have the form

$$A_0 = 0 \qquad A_a = A_a(\boldsymbol{x}) \tag{5.3}$$

Denoting $\psi_{\pm} = \frac{1}{2}(1 \mp \mathrm{i}\gamma_5)\psi$ we have

$$\psi = \psi_+ + \psi_- \qquad \psi_+ = \begin{pmatrix} \Phi \\ 0 \end{pmatrix} \qquad \psi_- = \begin{pmatrix} 0 \\ \xi \end{pmatrix} \tag{5.4}$$

where Φ and ξ are two component spinors, and 0 is the two component zero column. Expressing ψ_- via ψ_+ we come from (3.1) and (5.3) to the following equations:

$$(p_0^2 - m^2)\Phi = \hat{H}\Phi \qquad \hat{H} = \boldsymbol{\pi}^2 - e\boldsymbol{\sigma}\cdot\boldsymbol{H} \tag{5.5}$$

$$\xi = \frac{1}{m}(p_0 - \boldsymbol{\sigma}\cdot\boldsymbol{\pi})\Phi. \tag{5.6}$$

We will search for the SUSY of equation (5.5). The corresponding symmetries for (3.1) can be found using relations (5.4) and (5.6).

For the case of arbitrary vector potential $A(x)$ equation (5.5) admits the following symmetry operator (supercharge),

$$Q_1 = \boldsymbol{\sigma}\cdot\boldsymbol{\pi} \tag{5.7}$$

which satisfies the relation $Q_1^2 = \hat{H}$ and so commutes with the 'Hamiltonian' $\hat{H}$.

To find additional supercharges we suppose that the vector potential $A_a(x)$ satisfies one of the relations (3.3) where $\mu, \sigma = 1, 2, 3$. The corresponding equation (5.5) admits the following symmetries:

$$\Phi(t,\boldsymbol{x}) \to r_a\Phi(t,\boldsymbol{x}) \equiv \sigma_a\Phi(t,\theta_a\boldsymbol{x}) \tag{5.8a}$$

$$\Phi(t,\boldsymbol{x}) \to r_{ab}\Phi(t,\boldsymbol{x}) \equiv \varepsilon_{abc}\sigma_c\Phi(t,\theta_{ab}\boldsymbol{x}) \tag{5.8b}$$

$$\Phi(t,\boldsymbol{x}) \to r\Phi(t,\boldsymbol{x}) \equiv \Phi(t,\theta\boldsymbol{x}). \tag{5.8c}$$

Operators (5.8) satisfy the following relations (compare with (5.2)),

$$r_a^2 = r_{ab}^2 = r^2 = 1 \qquad \{r_{ab}, r_b\} = 0 \qquad \{r_a, r_b\} = 0 \qquad a \neq b \tag{5.9}$$

(now sum over b) and

$$\{r, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\} = \{r_a, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\} = 0 \qquad [r_{ab}, \boldsymbol{\sigma}\cdot\boldsymbol{\pi}] = 0 \tag{5.10}$$

which enable us to construct the second supercharges

$$Q_2 = \mathrm{i}r_a\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \qquad \text{and} \qquad Q_2 = \mathrm{i}r\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \tag{5.11}$$

for the cases (3.3a) and (3.3c), respectively.

Thus the corresponding equation (3.3) admits $N = 2$ SUSY.

If $A_a(\boldsymbol{x})$ satisfy two or more relations (3.3) simultaneously, then equation (5.5) admits extended SUSY. All non-equivalent possibilities are listed in the following formulae:

$$\begin{cases} A_a(\theta_b\boldsymbol{x}) = (1 - 2\delta_{ab})A_a(\boldsymbol{x}) \\ A_a(\theta_c\boldsymbol{x}) = (1 - 2\delta_{ac})A_a(\boldsymbol{x}) \end{cases} \tag{5.12}$$

$$Q_1 = \boldsymbol{\sigma}\cdot\boldsymbol{\pi} \qquad Q_2 = \mathrm{i}r_b\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \qquad Q_3 = \mathrm{i}r_c\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \qquad c \neq b \qquad b, c = 1, 2, 3$$

$$\begin{cases} A_a(\theta_{12}\boldsymbol{x}) = (1 - 2\delta_{a1} - 2\delta_{a2})A(\boldsymbol{x}) \\ A_a(\theta_{31}\boldsymbol{x}) = (1 - 2\delta_{a1} - 2\delta_{a3})A(\boldsymbol{x}) \end{cases} \tag{5.13}$$

$$Q_1 = \mathrm{i}r_{23}\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \qquad Q_2 = \mathrm{i}r_{31}\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \qquad Q_3 = \mathrm{i}r_{12}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}$$

$$\begin{cases} A_a(r_1\boldsymbol{x}) = (1 - 2\delta_{a1})A_a(\boldsymbol{x}) \\ A_a(r_2\boldsymbol{x}) = (1 - 2\delta_{2a})A_a(\boldsymbol{x}) \\ A_a(r_3\boldsymbol{x}) = (1 - 2\delta_{3a})A_a(\boldsymbol{x}) \end{cases} \tag{5.14}$$

$$Q_1 = \mathrm{i}r_1\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \qquad Q_2 = \mathrm{i}r_2\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \qquad Q_3 = \mathrm{i}r_3\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \qquad Q_4 = \boldsymbol{\sigma}\cdot\boldsymbol{\pi}.$$

Relations (5.12) define three classes of vector potentials A_μ (corresponding to different fixed values of b, c). Operators Q_a (5.12) and (5.13) realize $N = 3$ extended SUSY, while the corresponding operators (5.14) realize $N = 4$ extended SUSY.

It is necessary to note that the extended SUSY found in the above does not have direct connections with SUSY quantum field theory and cannot be used for a non-trivial extension of the Poincaré group. However, the symmetries (5.12)–(5.14) have rather non-trivial consequences in the quantum mechanical context, which consists of the specific degeneration of the corresponding energy spectra. Indeed, calculating *commutation* relations for the supercharges (5.12), we obtain

$$[Q_1, Q_2] = -\mathrm{i}r_b\hat{H} \qquad [Q_2, Q_3] = -\mathrm{i}r_{bc}\hat{H} \qquad [Q_3, Q_1] = \mathrm{i}r_c\hat{H}. \tag{5.15}$$

It follows from (5.9) and (5.15) that on the subspace of solutions of equation (5.5), corresponding to the non-zero eigenvalue E of the Hamiltonian $\hat{H}$, the operators

$$\hat{R}_{4k} = \frac{1}{\sqrt{E}}Q_k \qquad \hat{R}_{kl} = \frac{\mathrm{i}}{2|E|}[Q_k, Q_l] \tag{5.16}$$

satisfy relations (2.7) and (5.17):

$$\hat{R}_{kl}^2 = 1 \qquad \frac{1}{4!}\varepsilon_{klmn}\hat{R}_{kl}\hat{R}_{mn} = 3r_{ab}. \tag{5.17}$$

Eigenvalues of r_{ab} are equal to ± 1, and so operators (5.16) realize the representation $D(\frac{1}{2}, \frac{1}{2}) \oplus D(\frac{1}{2}, -\frac{1}{2})$ of the algebra $so(4)$.

In an analogous way, choosing the basis (5.16), we conclude that the supercharges (5.13) generate the same representation of the algebra $so(4)$, but the supercharges (5.14) generate the representation $D(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ of the algebra $so(5)$ (in the last case we have in (5.16) $k, l = 1, 2, 3, 5$). The corresponding commutation relations for operators (5.16) again can be expressed in form (2.7).

Thus, for any symmetry (5.12)–(5.14) each non-zero eigenvalue of the Hamiltonian $\hat{H}$ has fourfold degeneracy due to the hidden symmetry $so(4)$ or $so(5)$.

$N = 2$ and $N = 1$ SUSY aspects of the Dirac equation were discussed by a number of authors, refer, for example, to papers [20], surveys [12, 21] and the monograph [22]. We extend the list of problems generating this symmetry and find a class of potentials generating extended SUSY.

6. Reduction technique in supersymmetric quantum mechanics

The idea of diagonalizing a discrete symmetry in order to reduce the corresponding equation of motion can be applied to many problems in mathematical physics. Continuing the theme of SUSY, we apply this idea to one-dimensional SUSY quantum mechanics [19]. The corresponding equation of motion has the form

$$\mathrm{i}\frac{\partial}{\partial x_0}\psi = H\psi \tag{6.1}$$

where H is the Hamiltonian with matrix potential

$$H = \frac{1}{2}(\hat{p}^2 + W^2 + W'\sigma_3) \qquad \hat{p} = -\mathrm{i}\frac{\partial}{\partial x}. \tag{6.2}$$

Equation (6.1) admits specific symmetries (supercharges) of the form

$$Q = \frac{1}{2\sqrt{2}}(\sigma_2 + \mathrm{i}\sigma_1)(\hat{p} + \mathrm{i}W) \qquad \overline{Q} = \frac{1}{2\sqrt{2}}(\sigma_2 - \mathrm{i}\sigma_1)(\hat{p} - \mathrm{i}W) \tag{6.3}$$

which transform solutions into themselves and generate the following superalgebra (which is isomorphic to (5.1):

$$Q^2 = \overline{Q}^2 = 0 \qquad Q\overline{Q} + \overline{Q}Q = H$$
$$[Q, H] = [\overline{Q}, H] = 0. \tag{6.4}$$

Let us demonstrate that this superalgebra is reducible for odd superpotentials $W(x)$, i.e. for which $W(-x) = -W(x)$. Indeed, for W odd there exists the invariant operator

$$K = \sigma_3 p \tag{6.5}$$

$(p\psi(x) = \psi(-x))$ which commutes with supercharges Q and $\overline{Q}$. In order to diagonalize K we apply the operator

$$U = p_+ - \mathrm{i}\sigma_2 p_- \qquad p_\pm = \tfrac{1}{2}(1 \pm p) \tag{6.6}$$

so that

$$UKU^\dagger = \sigma_3. \tag{6.7}$$

The corresponding supercharges are transformed into the diagonal form

$$UQU^\dagger = Q' = \frac{\mathrm{i}}{2\sqrt{2}}[(1-\sigma_3)p_+ - (1+\sigma_3)p_-](\hat{p} + \mathrm{i}W)$$
$$U\overline{Q}U^\dagger = \overline{Q}' = -\frac{\mathrm{i}}{2\sqrt{2}}[(1-\sigma_3)p_- - (1+\sigma_3)p_+](\hat{p} - \mathrm{i}W)$$

i.e.

$$Q' = \begin{pmatrix} Q_+ & 0 \\ 0 & Q_- \end{pmatrix} \qquad \overline{Q}' = \begin{pmatrix} \overline{Q}_+ & 0 \\ 0 & \overline{Q}_- \end{pmatrix} \tag{6.8}$$

where

$$Q_+ = -\frac{\mathrm{i}}{\sqrt{2}}(\hat{p} + \mathrm{i}W)p_+ \qquad \overline{Q}_+ = \frac{\mathrm{i}}{\sqrt{2}}(\hat{p} - \mathrm{i}W)p_- \tag{6.9a}$$
$$Q_- = \frac{\mathrm{i}}{\sqrt{2}}(\hat{p} + \mathrm{i}W)p_- \qquad \overline{Q}_- = -\frac{\mathrm{i}}{\sqrt{2}}(\hat{p} - \mathrm{i}W)p_+. \tag{6.9b}$$

Thus supercharges (6.3) generate a reducible representation of the algebra (6.4) which is equivalent to a direct sum of representations (6.9*a*) and (6.9*b*). The corresponding Hamiltonians are of the form

$$H_+ = \tfrac{1}{2}(\hat{p}^2 + W^2 + W'p) \tag{6.10a}$$

and

$$H_- = \tfrac{1}{2}(\hat{p}^2 + W^2 - W'p). \tag{6.10b}$$

Operators (6.9*a*) and (6.10*a*) (as well as (6.9*b*) and (6.10*b*)) form a one-dimensional realization of SSQM, which has a very unique property: supercharges $Q_\pm$ and $\overline{Q}_\pm$ are not products of commutive bosonic and fermionic operators. As a consequence of this fact the spectra of superhamiltonians with familiar potentials differ from the corresponding spectra in standard realization of SSQM. For instance, if $W = \omega x$ then supercharges (6.9*b*) and Hamiltonian (6.10*b*) correspond to a specific version of the supersymmetric oscillator for which differences between eigenvalues are not equal to ω (compare with [19]) but to 2ω, whereas supercharges (6.9*a*) and Hamiltonian (6.10*a*) present a supersymmetric system with spontaneously broken supersymmetry (i.e. with a degenerated ground state).

In conclusion we notice that the $N = 2$ Wess–Zumino SSQM [23] with a superpotential being an odd complex function is also completely reducible too.

7. Conclusion

We present the extended Lie algebra formed by involutive symmetries of the Dirac equation and apply it to reduction of a number of problems connected with interaction of a spin-$\frac{1}{2}$ particle with an external field.

Such a reduction technique can be generalized for reduction of systems of ordinary differential equations as well as many other systems of partial differential equations, including nonlinear ones. We plan to outline the results of our investigations of these possibilities elsewhere.

The other interesting application of involutive symmetries is searching for exact SUSY for the Dirac equation. We demonstrate that in addition to the known class of systems with $N = 2$ SUSY this equation also generates extended supersymmetries. Moreover, the list of supersymmetric problems can be extended by including antilinear involutive symmetries.

The other goal of the present paper is to demonstrate that a wide class of realizations of SSQM is completely reducible. We obtain a one-dimensional representation of the Witten superalgebra (6.4) which can be extended to the case of multidimensional space of independent variables.

Finally, we note that the Weyl oscillators of section 5 are the simplest consistent examples of generalizations of the Dirac oscillator [17, 18]. Such generalizations for the cases of arbitrary spin multi-body systems are intensively discussed in the literature [24, 25].

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Appendix. Explicit form of reductions

Here we present in explicit form reductions of the Dirac equation which are possible if the vector potential A_μ satisfies one of the relations (3.3*a*–*c*) or (3.4*a*–*c*). To reduce (3.1) it is sufficient to diagonalize the corresponding symmetries (2.6) or (2.8). This can be done by using the operators

$$W_{4\nu} = \frac{1}{\sqrt{2}}(1 - \mathrm{i}\gamma_5 R_{4\nu}) \qquad \nu = 0, 1, 2, 3 \tag{A1a}$$

$$W_{\nu a} = \frac{1}{2}(1 + \mathrm{i}\gamma_5\gamma_a)(1 + \gamma_a R_{\nu a}) \qquad a = 1, 2, 3 \tag{A1b}$$

$$W_{5\nu} = \frac{1}{\sqrt{2}}(1 - \mathrm{i}\gamma_5 R_{5\nu}) \tag{A1c}$$

or

$$\begin{aligned} \overline{W}_{4a'} &= \frac{1}{2}(1 + \mathrm{i}\gamma_5\gamma_2)(1 - \mathrm{i}\gamma_2 B_{4a'}) \qquad a' = 1, 3 \\ \overline{W}_{42} &= \frac{1}{2}(1 + \mathrm{i}\gamma_5\gamma_0\hat{\theta}'_2 c)(1 + \gamma_0) \end{aligned} \tag{A2a}$$

$$\overline{W}_{5a} = \frac{1}{2}(1 + \gamma_4\gamma_0)(1 + \gamma_0 B_{5a}) \qquad a = 1, 2, 3 \tag{A2b}$$

$$\overline{W}_{0a} = \frac{1}{\sqrt{2}}(1 + \gamma_5 B_{0a}). \qquad \text{(A2}c\text{)}$$

In all the cases we have $W_{AB}R_{AB}W_{AB}^{\dagger} = \mathrm{i}\gamma_5$ or $\overline{W}_{AB}B_{AB}\overline{W}_{AB}^{\dagger} = \gamma_5$.

The transformations $\psi \to W_{AB}\psi$ reduce equation (3.1) to uncoupled subsystems of the following form:

$$\begin{aligned}
&[\pi_0 \mp (\boldsymbol{\sigma} \cdot \boldsymbol{\pi} - \mathrm{i}m\hat{\theta}_0)]\psi_{\pm} = 0 \\
&[\pi_0 \mp (\boldsymbol{\sigma} \cdot \boldsymbol{\pi} - m\hat{\theta}_a\sigma_a)]\psi_{\pm} = 0 \qquad \text{(A3}a\text{)}
\end{aligned}$$

$$\begin{aligned}
&[\sigma_a(\mp\pi_0 + m)\theta_{0a} + \boldsymbol{\sigma} \cdot \boldsymbol{\pi}]\psi_{\pm} = 0 \\
&[\pi_0 \mp \boldsymbol{\sigma} \cdot \boldsymbol{\pi} - \sigma_b m + (\mathrm{i}\sigma_a\theta_{ab} \pm \sigma_b)\pi_b]\psi_{\pm} = 0 \qquad \text{(A3}b\text{)}
\end{aligned}$$

$$\begin{aligned}
&[\pi_0 \mp (\boldsymbol{\sigma} \cdot \boldsymbol{\pi} + m\hat{\theta}'_0)]\psi_{\pm} = 0 \\
&[\pi_0 \mp \boldsymbol{\sigma} \cdot \boldsymbol{\pi} + \mathrm{i}m\hat{\theta}'_a\sigma_a]\psi_{\pm} = 0 \qquad \text{(A3}c\text{)}
\end{aligned}$$

or

$$\begin{aligned}
&(\hat{\theta}_2 c\pi_0 - \boldsymbol{\sigma} \cdot \boldsymbol{\pi} \pm \mathrm{i}m)\psi_{\pm} = 0 \\
&(\mp\pi_0 - \boldsymbol{\sigma} \cdot \boldsymbol{\pi} + \pi_{a'}(\sigma_{a'} \mp \mathrm{i}c\hat{\theta}'_{a'}) - c\hat{\theta}'_{a'}m)\psi_{\pm} = 0 \qquad a' = 1, 3 \qquad \text{(A4}a\text{)}
\end{aligned}$$

(no sums over repeated indices). Here $\psi_{\pm}$ are two-component wavefunctions, i.e. non-zero components of eigenvectors of the matrix

$$\begin{aligned}
&(\pm\pi_0 + \hat{\theta}_{a'}\sigma_{a'}\boldsymbol{\sigma} \cdot \boldsymbol{\pi} - m)\psi_{\pm} = 0 \\
&(\pm\pi_0 + c\hat{\theta}_2\boldsymbol{\sigma} \cdot \boldsymbol{\pi} - m)\psi_{\pm} = 0 \qquad \text{(A4}b\text{)}
\end{aligned}$$

$$\begin{aligned}
&(\pm\pi_0 - \boldsymbol{\sigma} \cdot \boldsymbol{\pi} - \mathrm{i}c\hat{\theta}_{03}\sigma_1 m)\psi_{\pm} = 0 \\
&(\pm\pi_0 - \boldsymbol{\sigma} \cdot \boldsymbol{\pi} - \mathrm{i}c\hat{\theta}_{01}\sigma_3 m)\psi_{\pm} = 0 \\
&(\pi_0 \mp \boldsymbol{\sigma} \cdot \boldsymbol{\pi} \mp c\hat{\theta}_{02}m)\psi_{\pm} = 0. \qquad \text{(A4}c\text{)}
\end{aligned}$$

γ_5 correspond to the eigenvalues ± 1.

Symmetry (4.8) commutes with operator L of (3.1) *iff* $A_\mu = 0$ and so can be used only for reduction of the free Dirac equation (2.1). The corresponding operator $W = (1/\sqrt{2})(1 + \mathrm{i}\gamma_5 C)$ diagonalizes symmetry (2.9) to the form γ_5 and reduces equation (2.1) to the following uncoupled subsystems:

$$(p_0 - \mu\boldsymbol{\sigma} \cdot \boldsymbol{p} + \mathrm{i}\sigma_2 cm)\psi_\mu = 0 \qquad \mu = \pm 1. \qquad \text{(A5)}$$

Imposing condition (4.5) on solutions of the first equation (A3*a*) and setting $A_\mu = 0$ we come to the equations proposed in [16].

If the vector potential A_μ has such parities that the corresponding Dirac equation (3.1) admits two commuting symmetries from the set (2.6) and (2.8) then we can reduce (3.1) to four uncoupled subsystems. If we find such a pair (S_1, S_2), then (S_1, S_1S_2) and (S_2, S_1S_2) are also sets of commuting symmetries equivalent to the set (S_1, S_2). Using (2.7) and (2.12), it is not difficult to write down the 51 non-equivalent pairs of commuting symmetries.

In other words, there are 51 possible reductions of the Dirac equation to four uncoupled subsystems by means of linear and antilinear involutive symmetries. The explicit form of these reductions can be found in analogy with the above.

$$\begin{aligned}
&\{R_{0a}, R_{bc}\}, \{R_{4a}, R_{bc}\}, \{R_{0a}, R_{4b}\}, \{R_{04}, R_{ab}\}, \{B_{4a}, R_{0b}\}, \\
&\{B_{0a}, R_{5b}\}, \{B_{5a}, R_{4b}\}, \{B_{5a}, R_{ab}\}, \{B_{4a}, R_{ab}\}, \{B_{4a}, R_{54}\}, \\
&\{B_{0a}, R_{40}\}, \{B_{0a}, R_{50}\}, \{B_{0a}, R_{50}\} \\
&a, b, c, = 1, 2, 3, a \neq b, a \neq c, b \neq c.
\end{aligned}$$

We notice that only two of the involutive symmetries of the Dirac equation commute with any Lorentz transformation, namely R and C given in (3.5) and (2.9), respectively. Consequently the corresponding reduced equations (3.9) and (4.19) are Lorentz invariant.

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